

STABILIZATION OF THE WATER-WAVE EQUATIONS WITH SURFACE TENSION

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ABSTRACT. This paper is devoted to the stabilization of the water-wave equations with surface tension through of an external pressure acting on a small part of the free surface. It is proved that the energy decays to zero exponentially in time, provided that the external pressure is given by the normal component of the velocity at the free surface multiplied by an appropriate cut-off function.

1. INTRODUCTION

Consider the incompressible Euler equations for a potential flow in a fluid domain located between with a free surface, two vertical walls and a flat bottom, which is at time t of the form

$$\Omega(t) = \{ (x, y) \in [0, L] \times \mathbb{R} : -h < y < \eta(t, x) \},$$

where L is the length of the basin, h is its depth and η , the free surface elevation, is an unknown function. We assume that the flow is irrotational so that the eulerian velocity field is the gradient of a potential function $\phi = \phi(t, x, y)$ satisfying

$$\begin{aligned} \Delta_{x,y} \phi &= 0 \quad \text{in } \Omega(t), \\ \partial_t \phi + \frac{1}{2} |\nabla_{x,y} \phi|^2 + P + gy &= 0 \quad \text{in } \Omega(t), \\ \phi_y &= 0 \quad \text{on } y = -h, \\ \phi_x &= 0 \quad \text{on } x = 0 \text{ or } x = L, \end{aligned} \tag{1.1}$$

where $P: \Omega \rightarrow \mathbb{R}$ is the pressure, $g > 0$ is the acceleration of gravity, $\nabla_{x,y} = (\partial_x, \partial_y)$ and $\Delta_{x,y} = \partial_x^2 + \partial_y^2$. Partial differentiations in space will be denoted by suffixes so that $\phi_x = \partial_x \phi$ and $\phi_y = \partial_y \phi$.

The water-wave equations are then given by two boundary conditions on the free surface: the classical kinematic boundary condition, describing the deformations of the domain,

$$\partial_t \eta = \sqrt{1 + \eta_x^2} \phi_n|_{y=\eta} = \phi_y(t, x, \eta(t, x)) - \eta_x(t, x) \phi_x(t, x, \eta(t, x)), \tag{1.2}$$

together with an equation expressing the balance of forces across the free surface:

$$P|_{y=\eta} = P_{ext} - \kappa H(\eta), \tag{1.3}$$

where $P_{ext} = P_{ext}(t, x)$ is the evaluation of the external pressure at the free surface, $\kappa \geq 0$ is the coefficient of surface tension and $H(\eta)$ is the curvature of the free surface:

$$H(\eta) = \partial_x \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right).$$

We are concerned with the problem with surface tension and assume that $\kappa > 0$.

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Introduce the energy $\mathcal{E} = \mathcal{E}(t)$, defined by

$$\mathcal{E} = \frac{g}{2} \int_0^L \eta^2 dx + \kappa \int_0^L \left(\sqrt{1 + \eta_x^2} - 1 \right) dx + \frac{1}{2} \int_0^L \int_{-h}^{\eta(t,x)} |\nabla_{x,y} \phi|^2 dy dx. \quad (1.4)$$

This is the sum of the gravitational potential energy, a surface energy due to stretching of the surface and the kinetic energy. Recall that the energy is conserved when there is no external pressure, which means that if $P_{ext} = 0$ then $\mathcal{E}(t) = \mathcal{E}(0)$ for all time. The stabilization problem for the water-wave equations consists in finding a pressure law, relating P_{ext} to the unknown (η, ψ) , such that:

- i) \mathcal{E} is decreasing and converges to zero exponentially in time;
- ii) $\partial_x P_{ext}$ is supported inside a small subset of $[0, L]$.

The stabilization problem corresponds to an important issue in the numerical analysis of water waves, namely the problem of damping outgoing waves in an absorbing zone surrounding the computational boundary. There is a huge literature about the absorption of the water-wave energy in a numerical wave tank and we refer the reader to [8, 9, 11, 12, 13, 18, 19, 20, 21, 22] and references therein. A popular choice is to assume that $P_{ext} = \chi \partial_t \eta$ for some cut-off function $\chi \geq 0$ supported¹ in $[L - \delta, L]$ for some $\delta > 0$. To explain this choice, we start by recalling that, as observed by Zakharov [31], the equations can be written in hamiltonian form. To do so, Zakharov introduced $\psi(t, x) = \phi(t, x, \eta(t, x))$, observed that the energy can be written as a function of (η, ψ) and verified that

$$\frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{E}}{\delta \psi}, \quad \frac{\partial \psi}{\partial t} = -\frac{\delta \mathcal{E}}{\delta \eta} - P_{ext}.$$

Using these equations, one deduces that

$$\frac{d\mathcal{E}}{dt} = \int \left(\frac{\delta \mathcal{E}}{\delta \eta} \frac{\partial \eta}{\partial t} + \frac{\delta \mathcal{E}}{\delta \psi} \frac{\partial \psi}{\partial t} \right) dx = - \int \frac{\partial \eta}{\partial t} P_{ext} dx, \quad (1.5)$$

and hence, if $P_{ext} = \chi \partial_t \eta$ with $\chi \geq 0$, we deduce that $d\mathcal{E}/dt \leq 0$. It is thus easily seen that the energy decays. However, it is much more complicated to prove that the energy converges exponentially to zero. To study this problem, we first need to pause to clarify the question since, in general, solutions of the water-wave equations do not exist globally in time (they might blow-up in finite time, see [10, 15]). Our goal is in fact to prove that there exists a constant C such that, if a regular solution exists on the time interval $[0, T]$, then

$$\mathcal{E}(T) \leq \frac{C}{T} \mathcal{E}(0). \quad (1.6)$$

Since the equation is invariant by translation in time, one can iterate this inequality. Consequently, if the solution exists on time intervals of size nT_0 with $T_0 \geq 2C$, then $\mathcal{E}(nT_0) \leq 2^{-n} \mathcal{E}(0)$, which is the desired exponential decay.

We studied a similar problem in [2] for the case without surface tension. Assuming that $\kappa = 0$ and defining P_{ext} by

$$\partial_x P_{ext} = \chi(x) \int_{-h}^{\eta(t,x)} \phi_x(t, x, y) dy, \quad (1.7)$$

¹We consider a damping located near $x = L$ only, and not also near $x = 0$, since we imagine that water waves are generated near $x = 0$. To do so, one could use also the variations of an external pressure. The latter problem is related to control theory and some references are given below.

where $\chi \geq 0$ is a cut-off function, we proved in [2] an inequality of the form

$$\mathcal{E}(T) \leq \frac{C(N)}{\sqrt{T}} \mathcal{E}(0), \quad (1.8)$$

where the constant $C(N)$ depends on the frequency localization² of the solution (η, ψ) . The fact that this constant must depend on the frequency localization can be easily understood by considering the linearized equations (see [2]). Somewhat surprisingly, it was possible to prove this result under some mild smallness condition on the solution which allows to consider truly nonlinear regimes. However, the main drawback is that one cannot control the frequency localization of the solutions in terms of the initial data, unless one makes strong smallness assumptions on these initial data (see §5 in [1]). In this paper, we will prove that one can exploit surface tension to improve the stabilization of water waves in three directions:

- i) the main improvement is that, under mild smallness assumptions on η , one can prove that $\mathcal{E}(T) \leq (C/T)\mathcal{E}(0)$ for some constant C depending only on the physical parameters g, κ, h, L . In particular, C is independent of the frequency localization. Another key point is that we will give an *explicit* bound for C . As a consequence, under explicit mild smallness assumptions on η , it is possible to use the induction argument mentioned above to obtain exponential decay of the energy. By combining this result with the local controllability result for small data proved by Alazard, Baldi and Han-Kwan in [3], this in turn implies a controllability result for non small data in large time (following an argument due to Dehman, Lebeau and Zuazua [17]).
- ii) The second important improvement is that we are able to consider the case when $P_{ext} = \chi \partial_t \eta$ instead of assuming that P_{ext} is given by (1.7). This is important for applications since, as mentioned above, this feedback law is widely used in the numerical analysis of water waves.
- iii) Eventually we obtain an inequality of the form (1.6) with a factor $1/T$ instead of the factor $1/\sqrt{T}$ of (1.8).

2. MAIN RESULT

Following Zakharov [31] and Craig–Sulem [16] (see also [4]), one reduces the water-wave equations to a system on the free surface. To do so, remarking that the velocity potential ϕ is harmonic and satisfies a Neumann boundary condition on the walls and the bottom, we see that ϕ is fully determined by its evaluation at the free surface. We thus work below with

$$\psi(t, x) := \phi(t, x, \eta(t, x)),$$

and introduce the Dirichlet to Neumann operator, denoted by $G(\eta)$, relating ψ to the normal derivative of the potential by

$$G(\eta)\psi = \sqrt{1 + \eta_x^2} \phi_n|_{y=\eta} = (\phi_y - \eta_x \phi_x)|_{y=\eta}.$$

Then, it follows from (1.2) that $\partial_t \eta = G(\eta)\psi$ and (1.1)–(1.3) implies that

$$\begin{aligned} \partial_t \psi + g\eta + N(\eta)\psi - \kappa H(\eta) &= -P_{ext}, \quad \text{where} \\ N(\eta)\psi &= \mathcal{N}|_{y=\eta} \quad \text{with} \quad \mathcal{N} = \frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 + \eta_x \phi_x \phi_y, \end{aligned} \quad (2.1)$$

²The quantity N measuring the frequency localization is of the ratio of two Sobolev norms. One can think of the ratio $\|u\|_{H^1} / \|u\|_{L^2}$, which is proportional to N for a typical function oscillating at frequency N , like $u(x) = \cos(2\pi Nx/L)$.

where recall that $H(\eta) = \partial_x \left(\eta_x / \sqrt{1 + \eta_x^2} \right)$. One can check that

$$N(\eta)\psi = \frac{1}{2}\psi_x^2 - \frac{1}{2} \frac{(G(\eta)\psi + \eta_x\psi_x)^2}{1 + \eta_x^2}.$$

With the above notations, the water-wave system reads

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + N(\eta)\psi - \kappa H(\eta) = -P_{ext}. \end{cases} \quad (2.2)$$

In this paper we consider solutions (η, ψ) of (2.2) which are periodic in x , with period $2L$ and we further assume that these solutions are even in x . As explained below, the reason why we are making these symmetry assumptions is that this setting corresponds to the setting discussed in the introduction. That is the case where the water waves are propagating in a bounded container with length L , having vertical walls located on $x = 0$ and $x = L$, so that at time t the fluid domain is

$$\Omega(t) = \{ (x, y) \in [0, L] \times \mathbb{R} : -h < y < \eta(t, x) \}. \quad (2.3)$$

Definition 2.1 (Regular solutions). *Let $L > 0$, $T > 0$. Denote by \mathbb{T} the torus $\mathbb{R}/(2L\mathbb{Z})$ and, for any $\sigma \in \mathbb{R}$, denote by $H^\sigma(\mathbb{T})$ the Sobolev space of order σ of $2L$ -periodic functions. We say that (η, ψ, P_{ext}) is a regular solution of (2.2) defined on $[0, T]$ provided that the following three conditions are satisfied:*

i) there exists a real number $s > 5/2$ such that

$$\begin{aligned} \eta &\in C^0([0, T]; H^{s+\frac{1}{2}}(\mathbb{T})), \quad \psi \in C^0([0, T]; H^s(\mathbb{T})), \\ P_{ext} &\in C^0([0, T]; H^{s-1}(\mathbb{T})); \end{aligned}$$

ii) η , ψ and P_{ext} are even in x ;

iii) for all t in $[0, T]$,

$$\inf_{x \in [-L, L]} \eta(t, x) \geq -\frac{h}{2}, \quad \int_{-L}^L \eta(t, x) dx = 0, \quad \int_{-L}^L P_{ext}(t, x) dx = 0. \quad (2.4)$$

Remark 2.2. *i) We can make without loss of generality an assumption on the mean value of η since it is a conserved quantity.*

ii) Let us recall from [5] an argument which shows that this setting allows us to consider water waves in a bounded container. Consider a regular solution (η, ψ, P_{ext}) of (2.2) defined on $[0, T]$. The assumption that η and ψ are even and periodic with period $2L$ implies that ϕ is also even and periodic in x . We deduce that ϕ_x is odd in x and hence $\phi_x(t, x, y) = 0$ whenever $x = 0$ or $x = L$. In particular, the normal component of the velocity satisfies the solid wall boundary condition (that is $v \cdot n = 0$) on both the bottom and the lateral walls ($\{x = 0\}$ or $\{x = L\}$). Consequently, as illustrated below, we obtain a solution to the problem in a bounded container by considering the restrictions of η to $0 \leq x \leq L$ and the restriction of ϕ to Ω given by (2.3) (see Figure 1 below).

We denote by \mathcal{H} the energy

$$\mathcal{H} = \frac{g}{2} \int_{-L}^L \eta^2 dx + \kappa \int_{-L}^L \left(\sqrt{1 + \eta_x^2} - 1 \right) dx + \frac{1}{2} \int_{-L}^L \int_{-h}^{\eta(t, x)} |\nabla_{x, y} \phi|^2 dy dx. \quad (2.5)$$

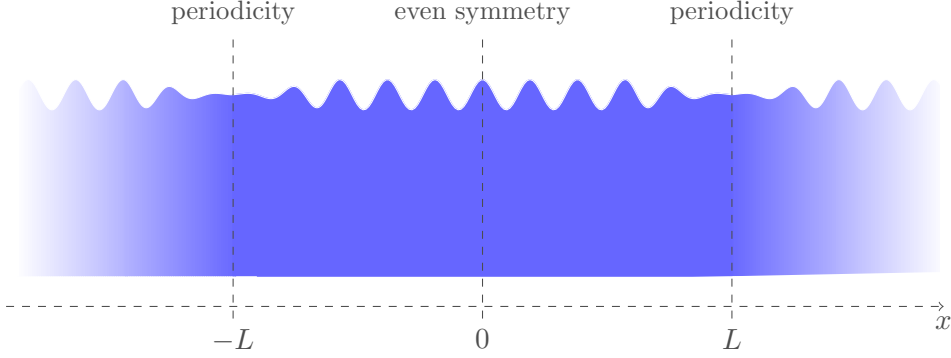


FIGURE 1. Periodic and even solutions.

Remark 2.3. Since η and ϕ are even and periodic in x with period $2L$, we deduce that $\mathcal{H} = 2\mathcal{E}$ where \mathcal{E} is as defined in the introduction (see (1.4)), that is

$$\mathcal{E} = \frac{g}{2} \int_0^L \eta^2 dx + \kappa \int_0^L \left(\sqrt{1 + \eta_x^2} - 1 \right) dx + \frac{1}{2} \int_0^L \int_{-h}^{\eta(t,x)} |\nabla_{x,y} \phi|^2 dy dx.$$

We now define the pressure law. We find convenient to impose that the mean value³ of P_{ext} is zero, so we set

$$P_{ext}(t, x) = \chi(x) \partial_t \eta(t, x) - \frac{1}{2L} \int_{-L}^L \chi(x) \partial_t \eta(t, x) dx, \quad (2.6)$$

for some cut-off function $\chi \geq 0$ to be determined. In order to exploit a cancellation in the proof, we choose χ as follows.

Definition 2.4. Fix $\delta > 0$ and consider a $2L$ -periodic C^∞ cut-off function φ , satisfying $0 \leq \varphi \leq 1$ and such that

$$\varphi(x) = \varphi(-x), \quad x\varphi'(x) \leq 0 \text{ for } x \in [-L, L], \quad \varphi(x) = \begin{cases} 1 & \text{if } x \in [0, L - \delta], \\ 0 & \text{if } x \in \left[L - \frac{\delta}{2}, L \right]. \end{cases}$$

We successively set

$$m(x) = x\varphi(x),$$

and

$$\chi(x) = 1 - m_x(x).$$

³In the introduction, we mentioned that the pressure law used in the literature is simply $P_{ext} = \chi \partial_t \eta$. Let us give two arguments showing that one can add a time dependent function to the pressure. The first observation is that, since $\frac{d}{dt} \int \eta dx = 0$ (conservation of the volume), for any function $F = F(t)$ depending only on time we have $\int F \partial_t \eta dx = 0$ so $\int \frac{\partial \eta}{\partial t} P_{ext} dx = \int \frac{\partial \eta}{\partial t} (P_{ext} + F) dx$. Consequently, the argument given above in the introduction to deduce that the energy decays (see (1.5)) still applies when P_{ext} is replaced by $P_{ext} + F(t)$. The second observation is that, when P_{ext} is given by (2.6), the pressure is not supported inside a small subset of $[-L, L]$ but this is not an obstruction since we only require that its spatial derivative is supported inside a small subset of $[-L, L]$. Indeed, the only quantity which is physically meaningful is the velocity which is the gradient of the velocity potential. In other words, we could also work with $P_{ext} = \chi \partial_t \eta$ by modifying the Bernouilli's constant in the equation for ψ .

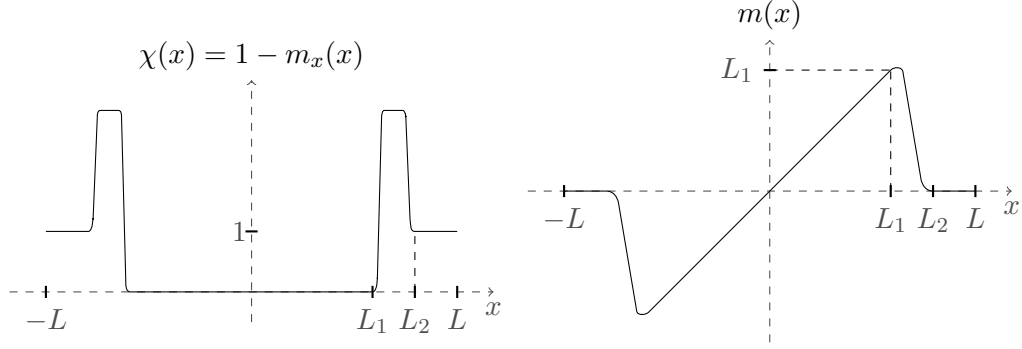


FIGURE 2. The functions χ and m , here $L_1 = L - \delta$ and $L_2 = L - \delta/2$.

Remark 2.5. Let m be as the given by Figure 2 and define φ by $\varphi(x) = m(x)/x$. One immediately see that

$$0 \leq \varphi \leq 1, \quad \varphi(x) = \varphi(-x), \quad \varphi(x) = \begin{cases} 1 & \text{if } x \in [0, L - \delta], \\ 0 & \text{if } x \in \left[L - \frac{\delta}{2}, L\right]. \end{cases}$$

It remains to check that $x\varphi'(x) \leq 0$ for $x \in [-L, L]$. We have $x\varphi'(x) = f(x)/x^2$ with $f(x) = xm'(x) - m(x)$. Set $L' = (L_1 + L_2)/2$, so that m is concave on $[0, L']$ and convex on $[L', L]$. Observe that $f(0) = 0$ and $f'(x) = xm''(x) \leq 0$ for $x \in [0, L']$ since m is concave there, which proves that $f(x) \leq 0$ on $[0, L']$. On the other hand, for $x \geq L'$, one has $m'(x) \leq 0$ and $m(x) \geq 0$ so one obviously get $f(x) \leq 0$. This proves that $x\varphi'(x) \leq 0$ on $[0, L]$. Since $x\varphi'(x)$ is even in x , we deduce that $x\varphi'(x) \leq 0$ on $[-L, L]$.

Here is our main result. Recall that κ is the surface tension coefficient, g is the acceleration of gravity, h is the depth of the fluid domain at rest and L the width of the basin (half of the period).

Theorem 2.6. *Let m, χ be as given by Definition 2.4. Assume that $\kappa > 0$ and*

$$\kappa \sup_{[-L, L]} m_{xx}(x)^2 \leq g. \quad (2.7)$$

Then there exists a positive constant C , depending only on the physical parameters κ, g, h, L , such that the following result holds. Let $T > 0$ and consider a regular solution (η, ψ, P_{ext}) defined on the time interval $[0, T]$ with P_{ext} as given by (2.6). Set

$$\rho(t, x) = (m(x) - x)\eta_x(t, x) + \frac{9}{4}\eta(t, x) - \frac{1}{2}m_x(x)\eta(t, x).$$

If, for all $(t, x) \in [0, T] \times [-L, L]$,

- i) $\rho(t, x) \geq -\frac{h}{4}, \quad |\rho_x(t, x)| < \frac{1}{4},$*
- ii) $\int_{-L}^L (1 - m_x(x))\eta(t, x) dx \leq \frac{hL}{3}, \quad |m_x(x)| |\eta_x(t, x)|^2 \leq 2, \quad |\eta_x(t, x)| \leq 1,$*

then one has the estimate

$$\mathcal{H}(T) \leq \frac{C}{T} \mathcal{H}(0).$$

Remark 2.7. *i)* One can choose m as in Figure 2 so that $|m_{xx}| \leq 8\delta^{-1} + 32L\delta^{-2}$. Then (2.7) is satisfied provided that $\kappa \leq (8\delta^{-1} + 32L\delta^{-2})^{-2}g$.

ii) An important remark is that the constant C can be given by an explicit formula in terms of κ, g, h, L . Namely, C is given by $C = 8K$ where K is as defined in (4.19) (taking $\lambda = 1$ in (4.19)).

To conclude this introduction, we outline the main steps of the proof and also comment on related questions about the study of the Cauchy problem or the controllability of the water-wave equations.

2.1. About the Cauchy problem. As already mentioned, Theorem 2.6 implies that the energy converges exponentially to zero in the following sense: let $T_0 = 2C$ where C is as given by Theorem 2.6 and consider a regular solution (η, ψ, P_{ext}) defined on the time interval $[0, nT_0]$, then one has

$$\mathcal{H}(nT_0) \leq 2^{-n}\mathcal{H}(0).$$

This remark raises a question about the Cauchy problem, namely to prove that one can obtain regular solutions on arbitrary large time intervals. We will not study this problem in this paper because it involves different tools. Here, we only state a result, which will be proved in a separate paper, implying that regular solutions exist on large time intervals, under a smallness assumption on the initial data.

Consider the equations

$$\begin{cases} \partial_t \eta = G(\eta)\psi, \\ \partial_t \psi + g\eta + N(\eta)\psi - \kappa H(\eta) = -P_{ext}, \end{cases} \quad (2.8)$$

where P_{ext} is given by

$$P_{ext}(t, x) = \chi(x)\partial_t \eta - \frac{1}{2L} \int_{-L}^L \chi(x)\partial_t \eta(t, x) dx, \quad (2.9)$$

for some function $\chi \geq 0$ (the following result holds for any function χ , and in particular we do not need here to assume that χ can be written as $1 - m_x$ for some multiplier m of the form $m(x) = x\varphi(x)$). Consider now an initial data (η_0, ψ_0) in $H^{s+\frac{1}{2}}(\mathbb{T}) \times H^s(\mathbb{T})$ for some $s > 5/2$, $2L$ -periodic and even, and set

$$m = \|\eta_0\|_{H^{s+\frac{1}{2}}(\mathbb{T})} + \|\psi_0\|_{H^s(\mathbb{T})}.$$

Then the Cauchy problem for the system (2.8)–(2.9) has a unique regular solution (which means that (η, ψ, P_{ext}) satisfies the three properties of Definition 2.1) on the time interval $[0, c_*/m]$ for some positive constant c_* depending only on s . Moreover, one has the estimate

$$\sup_{t \in [0, c_*/m]} \left\{ \|\eta(t)\|_{H^{s+\frac{1}{2}}(\mathbb{T})} + \|\psi(t)\|_{H^s(\mathbb{T})} \right\} \leq 2m.$$

2.2. Application to the controllability of the water-wave equations. There are many results about the controllability or the stabilization of linear or nonlinear equations describing water waves in some asymptotic regimes like Benjamin-Ono, KdV or nonlinear Schrödinger equations (we refer the reader to the book of Coron [14]). However, one cannot easily adapt these studies to the water-wave system (2.8) since the latter system is quasi-linear (instead of semi-linear) and since it is a pseudo-differential system, involving the Dirichlet-Neumann operator which is nonlocal and also depends nonlinearly on the unknown. The first results about the possible applications of control theory to the water-wave equations are due to Reid

and Russell [30] and Reid [28, 29], who studied the linearized equations at the origin (see also Miller [27], Lissy [26] and Biccari [7] for other control results about dispersive equations involving a fractional Laplacian). Alazard, Baldi and Han-Kwan proved in [3] the first controllability result for the nonlinear water-wave equations with surface tension, namely a controllability result in arbitrarily small time, under a smallness assumption on the size of the data.

Theorem 2.8 (Alazard, Baldi, Han-Kwan, from [3]). *Assume that $\kappa > 0$. Let $T > 0$ and consider a non-empty open subset $\omega \subset \mathbb{T}$. There exist σ large enough and a positive constant ε_c small enough such that, for any two pairs of functions (η_{in}, ψ_{in}) , $(\eta_{final}, \psi_{final})$ in $H^{\sigma+\frac{1}{2}}(\mathbb{T}) \times H^\sigma(\mathbb{T})$ satisfying*

$$\|\eta_{in}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{in}\|_{H^\sigma} < \varepsilon_c, \quad \|\eta_{final}\|_{H^{\sigma+\frac{1}{2}}} + \|\psi_{final}\|_{H^\sigma} < \varepsilon_c,$$

and such that $\int \eta_{in} dx = \int \eta_{final} dx = 0$, there exists P_{ext} in $C^0([0, T]; H^\sigma(\mathbb{T}))$, supported in $[0, T] \times \omega$, such that the Cauchy problem for (2.2) has a unique solution

$$(\eta, \psi) \in C^0([0, T]; H^{\sigma+\frac{1}{2}}(\mathbb{T}) \times H^\sigma(\mathbb{T})),$$

and the solution (η, ψ) satisfies $(\eta|_{t=T}, \psi|_{t=T}) = (\eta_{final}, \psi_{final})$.

The smallness condition in [3] comes from various technical arguments and it is possibly a strong assumption. In sharp contrast, Theorem 2.6 holds under explicit and mild smallness conditions on η . One can combine both results to obtain a controllability result for larger initial data in large time. This strategy was used by Dehman-Lebeau-Zuazua [17] and Laurent [24, 25] for other wave equations. The idea is to proceed in two steps: firstly one steers the solution close to zero by taking as a control the stabilization term so that it is possible, in a second step, to use the local controllability near zero to steer the solution to zero. This proves a null controllability result. By using another standard argument, exploiting the fact that the equation is reversible in time, one deduces the wanted controllability result from this null controllability result.

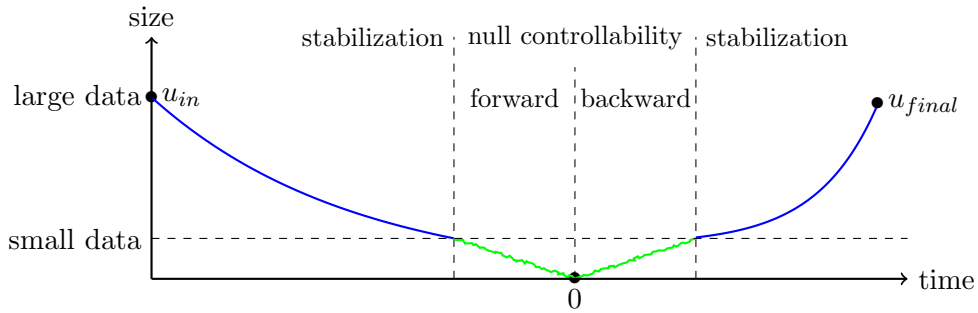


FIGURE 3. The Dehman-Lebeau-Zuazua strategy (cf [17]).

2.3. Strategy of the proof. To prove Theorem 2.6, our goal is to prove that there exists a positive constant C such that

$$\int_0^T \mathcal{H}(t) dt \leq C\mathcal{H}(0). \quad (2.10)$$

Since \mathcal{H} is a decreasing function, this will imply the wanted inequality

$$\mathcal{H}(T) \leq \frac{1}{T} \int_0^T \mathcal{H}(t) dt \leq \frac{C}{T} \mathcal{H}(0).$$

The proof of (2.10) is in two steps.

First step. Here we derive an exact *identity* which involves the integral in time of the energy. This approach is not new. It was already performed in our previous works [1, 2] and was based on several tools: the multiplier technique (with the multiplier $m(x)\partial_x$ for some function m to be determined), the Craig-Sulem-Zakharov reduction to an hamiltonian system on the boundary, a Pohozaev identity for the Dirichlet to Neumann operator, previous results about the Cauchy problem and computations guided by the analysis done by Benjamin and Olver of the conservation laws for water waves (cf [6]). We need here to adapt this analysis to the present context with surface tension, which requires new ideas. Indeed, compared to the case without surface tension, we will have to handle many remainder terms. In the end, instead of stating an identity (as we did for the case without surface tension), we will obtain an inequality.

Consider a smooth function $m = m(x)$ with $m(L) = 0$, and set

$$\zeta = \partial_x(m\eta) + \frac{3}{2}(1 - m_x)\eta - \frac{1}{4}\eta,$$

our first main task will be to derive the following inequality

$$\begin{aligned} \frac{1}{4} \int_0^T \mathcal{H}(t) dt &\leq O + W + B - I \quad \text{where} \\ I &:= \frac{h}{4} \int_0^T \int_{-L}^L \phi_x(t, x, -h)^2 dx dt, \\ O &:= \int_0^T \int_{-L}^L \left(\frac{3}{2}(1 - m_x)\tilde{\psi} + (x - m)\psi_x \right) G(\eta)\psi dx dt, \\ W &:= - \int_0^T \int_{-L}^L P_{ext} \zeta dx dt, \\ B &:= \int_{-L}^L \zeta(0, x)\tilde{\psi}(0, x) dx - \int_{-L}^L \zeta(T, x)\tilde{\psi}(T, x) dx, \end{aligned} \tag{2.11}$$

where $\tilde{\psi}$ is equal to ψ minus its average on $[-L, L]$. These four quantities play different roles. Their key properties are the following:

- $I \geq 0$ and hence (2.11) gives a bound for the horizontal component of the velocity at the bottom (this plays a key role to control the velocity in terms of the pressure, see (2.14)).
- W is the only term which involves the pressure.
- O corresponds to an *observation*, this means that this term depends only on the behavior of the solutions near $\{x = -L\}$ or $\{x = L\}$ when m is as given by Definition 2.4. Indeed, $x - m$ and $1 - m_x$ vanish when $x \in [-L + \delta, L - \delta]$ by definition of m .
- B is not an integral in time, by contrast with the other terms and, in addition, it is easily estimated by $K\mathcal{H}(0)$.

Second step. The goal of the second step is to deduce the wanted result (2.10) from the identity (2.11) proved in the first step. To do so, it is sufficient to prove

that there exists a constant K depending only on g, κ, h, L such that

$$O + W + B - I \leq K\mathcal{H}(0) + a \int_0^T \mathcal{H}(t) dt \quad \text{for some } a < \frac{1}{4}. \quad (2.12)$$

Indeed, by combining (2.11) and (2.12) one obtains that

$$\int_0^T \mathcal{H}(t) dt \leq \frac{K}{1/4 - a} \mathcal{H}(0),$$

which is the wanted result (2.10). To prove (2.12), one estimates separately the terms B, W, O .

The estimate of the term B is easy so we merely explain how to estimate W, O . To explain the first ingredient of the proof, recall that the hamiltonian structure of the equation implies that

$$\frac{d}{dt} \mathcal{H}(t) = - \int_{-L}^L P_{ext} \partial_t \eta dx.$$

Since $\partial_t \eta = G(\eta)\psi$, one deduces that

$$\int_0^T \int_{-L}^L \chi(\partial_t \eta)^2 dx dt = \int_0^T \int_{-L}^L \chi(G(\eta)\psi)^2 dx dt = \int_0^T \int_{-L}^L P_{ext} \partial_t \eta dx dt \leq \mathcal{H}(0).$$

This gives an estimate for the L^2 -norm of $\chi \partial_t \eta$, which is the main contribution to the definition of P_{ext} (see (2.9)). A more tricky inequality, relying on the special choice $\chi(x) = 1 - m_x(x)$, is that

$$- \int_0^T \left(\int_{-L}^L \chi \partial_t \eta dx \right) \int_{-L}^L \zeta dx dt \leq \frac{L}{g} \sup_{[-L, L]} (1 - m_x)^2 \mathcal{H}(0). \quad (2.13)$$

By combining the above inequalities, we will be able to estimate the term W . The main difficulty is to bound the observation term O , and in particular to estimate the contribution of

$$\iint (x - m) \psi_x G(\eta) \psi dx dt.$$

Using the Cauchy-Schwarz inequality, the key point is to estimate the L^2 -norm of ψ_x in terms of the two quantities which are under control, that is the integral of $\chi(G(\eta)\psi)^2$ and the positive term I . In this direction, we will prove the following inequality, which is of independent interest: there exists a constant A , depending only on $\|\eta_x\|_{L^\infty}$, such that

$$\begin{aligned} \int_{-L}^L \chi \psi_x^2 dx &\leq A \int_{-L}^L \chi(G(\eta)\psi)^2 dx + A \int_{-L}^L \phi_x^2|_{y=-h} dx \\ &\quad - A \int_{-L}^L \int_{-h}^{\eta(t,x)} \chi_x \phi_x \phi_y dy dx. \end{aligned} \quad (2.14)$$

Notice that this result holds in fact for any smooth cut-off function χ and one can also take $\chi = 1$. In this case this inequality simplifies since the last term in the right-hand side vanishes. This gives a way to control the L^2 -norm of ψ_x by the L^2 -norm of $G(\eta)\psi$. When η is smooth, this can be obtained by delicate commutator estimates. Here we will give a simple proof which applies for any Lipschitz domain.

3. AN INTEGRAL INEQUALITY

As explained in Section 2.3, the first key step is to obtain an inequality which involves the integral in time of the energy.

Proposition 3.1. *Let $m \in C^\infty(\mathbb{T})$ be a smooth $2L$ -periodic function which is odd and such that $m(L) = 0$ (in particular $m(-L) = m(0) = 0$). Consider a regular solution of the water-wave system defined on the time interval $[0, T]$ (see Definition 2.1) and set*

$$\zeta = \partial_x(m\eta) + \frac{3}{2}(1 - m_x)\eta - \frac{1}{4}\eta, \quad \rho = \zeta + \eta - x\eta_x.$$

Assume that, for all time $t \in [0, T]$ and all $x \in [-L, L]$, the following assumptions hold:

- i) $\rho(t, x) \geq -\frac{h}{4}, \quad |\rho_x(t, x)| < \frac{1}{4},$
- ii) $\int_{-L}^L (1 - m_x(x))\eta(t, x) dx \leq \frac{hL}{3}, \quad |m_x(x)| |\eta_x(t, x)|^2 \leq 2,$
- iii) $\kappa m_{xx}(x)^2 \leq g, \quad m_x(x) \leq 1.$

Then,

$$\begin{aligned} \frac{1}{4} \int_0^T \mathcal{H}(t) dt + \frac{h}{4} \int_0^T \int_{-L}^L \phi_x(t, x, -h)^2 dx dt \\ \leq \int_0^T \int_{-L}^L \left(\frac{3}{2}(1 - m_x)\tilde{\psi} + (x - m)\psi_x \right) G(\eta)\psi dx dt \\ - \int_0^T \int_{-L}^L P_{ext} \zeta dx dt - \int_{-L}^L \zeta \tilde{\psi} dx \Big|_0^T, \end{aligned} \quad (3.1)$$

with

$$\tilde{\psi}(t, x) = \psi(t, x) - \frac{1}{2L} \int_{-L}^L \psi(t, x) dx, \quad (3.2)$$

where $\int_{-L}^L f dx \Big|_0^T$ is a shorthand notation for $\int_{-L}^L f(T, x) dx - \int_{-L}^L f(0, x) dx$.

Remark 3.2. This result holds for any regular solutions (that is, without requiring that P_{ext} is related to (η, ψ)). Also, the assumptions on m are less restrictive than the ones given by Definition 2.4.

Proof. The proof follows the analysis in [1, 2]. The main novelty is that we are now able to take into account surface tension.

Notation. We write simply

$$\int dx, \quad \int dy, \quad \int dt,$$

as shorthand notations for, respectively,

$$\int_{-L}^L dx, \quad \int_{-h}^{\eta(t, x)} dy, \quad \int_0^T dt.$$

Then, $\iint f dx dt = \int_0^T \int_{-L}^L f(t, x) dx dt$ together with similar conventions for other integrals.

The proof is in four steps. The first step is to use the multiplier method to obtain an identity. We use the following variant of the multiplier method (as introduced in our previous paper [1]): we set

$$A := \iint \{(\partial_t \eta)(m \partial_x \psi) - (\partial_t \psi)(m \partial_x \eta)\} \, dx \, dt, \quad (3.3)$$

and compute A in two different ways.

Lemma 3.3. *There holds,*

$$\begin{aligned} & \iint m_x \left(-\eta \partial_t \psi - \frac{g}{2} \eta^2 - \kappa \frac{1}{\sqrt{1 + \eta_x^2}} \right) \, dx \, dt + \int F(t) \, dt \\ &= - \int \partial_x(m\eta) \psi \, dx \Big|_0^T - \iint P_{ext} m \eta_x \, dx \, dt, \end{aligned}$$

where

$$F(t) = \int (G(\eta) \psi) m \psi_x \, dx + \int (N(\eta) \psi) m \eta_x \, dx. \quad (3.4)$$

Proof. Since $m(-L) = m(L) = 0$, integrating by parts in space and time, we find that A (as given by (3.3)) satisfies

$$A = - \int \partial_x(m\eta) \psi \, dx \Big|_0^T + \iint m_x \eta \partial_t \psi \, dx \, dt. \quad (3.5)$$

Now, we compute A by replacing $\partial_t \eta$ and $\partial_t \psi$ by the expressions given by System (2.2). We find that

$$A = \iint \left(P_{ext} + g\eta - \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right) \right) m \eta_x \, dx \, dt + \int F(t) \, dt, \quad (3.6)$$

where F is given by (3.4). Since $m(-L) = m(L) = 0$, integrating by parts, we obtain

$$- \int g \eta m \eta_x \, dx = \frac{1}{2} \int g m_x \eta^2 \, dx.$$

Similarly, integrating by parts in x , one can write

$$\begin{aligned} - \iint \partial_x \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right) m \eta_x \, dx &= \int m_x \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} \, dx + \int \frac{\eta_x \eta_{xx}}{\sqrt{1 + \eta_x^2}} m \, dx \\ &= \int m_x \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} + \int m \partial_x \sqrt{1 + \eta_x^2} \, dx \\ &= \int m_x \left(\frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} - \sqrt{1 + \eta_x^2} \right) \, dx \\ &= - \int m_x \frac{1}{\sqrt{1 + \eta_x^2}} \, dx. \end{aligned}$$

By combining this identity with (3.5) and (3.6), we obtain the wanted result. \square

Our next step is to transform the previous identity to make appear a quantity which controls the potential energy. When there is no surface tension, it is easy to do so. By contrast, with surface tension, we obtain an identity with a remainder (see R_1 below) which will be estimated later on.

Lemma 3.4. *There holds*

$$\begin{aligned}
& \iint \left(\frac{g}{2} \eta^2 + \frac{3}{2} \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \right) dx dt + \int R_1(t) dt \\
&= \iint (1 - m_x) \left(\frac{g}{2} \eta^2 + \frac{3}{2} \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \right) dx dt \\
&\quad - \int \partial_x(m\eta) \psi dx \Big|_0^T - \iint P_{ext} \partial_x(m\eta) dx dt \\
&\quad - \iint (G(\eta)\psi)(m\psi_x) dx dt - \iint (N(\eta)\psi) \partial_x(m\eta) dx dt,
\end{aligned} \tag{3.7}$$

where R_1 is given by

$$R_1(t) = \kappa \int m_{xx} \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} dx + \kappa \int m_x \left(1 - \frac{1}{2} \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} - \frac{1}{\sqrt{1+\eta_x^2}} \right) dx. \tag{3.8}$$

Proof. Directly from the equation for ψ , we have

$$\begin{aligned}
& \iint -m_x \eta \partial_t \psi dx dt = \\
&= \iint m_x \left(g \eta^2 + \eta N(\eta) \psi - \kappa \eta \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) + \eta P_{ext} \right) dx dt.
\end{aligned}$$

Noticing that η_x vanishes for $x = -L$ or $x = L$ (since η is C^1 , even and $2L$ -periodic), we can integrate by parts in the term involving the surface tension, to obtain

$$\begin{aligned}
& \iint -m_x \eta \partial_t \psi dx dt = \iint (g m_x \eta^2 + m_x \eta N(\eta) \psi + m_x \eta P_{ext}) dx dt \\
&\quad + \iint \kappa m_x \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} dx dt + \iint \kappa m_{xx} \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} dx dt.
\end{aligned}$$

By combining this identity with Lemma 3.3, we deduce that

$$\begin{aligned}
& \iint m_x \left(\frac{g}{2} \eta^2 + \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} - \kappa \frac{1}{\sqrt{1+\eta_x^2}} \right) dx dt + \kappa \iint m_{xx} \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} dx dt \\
&= - \iint P_{ext} \partial_x(m\eta) dx dt - \int \partial_x(m\eta) \psi dx \Big|_0^T \\
&\quad - \iint (G(\eta)\psi)(m\psi_x) dx dt - \iint (N(\eta)\psi) \partial_x(m\eta) dx dt.
\end{aligned} \tag{3.9}$$

Since $m(-L) = m(L) = 0$, we have $\int m_x dx = 0$ which implies that

$$\begin{aligned}
& - \iint m_x \frac{1}{\sqrt{1+\eta_x^2}} dx dt = \\
&= \iint m_x \left(\frac{1}{2} \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} + \left[-\frac{1}{2} \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} - \frac{1}{\sqrt{1+\eta_x^2}} + 1 \right] \right) dx dt.
\end{aligned}$$

By reporting this identity in (3.9), we conclude that

$$\begin{aligned}
& \iint m_x \left(\frac{g}{2} \eta^2 + \frac{3\kappa}{2} \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \right) dx dt + \int R_1 dt = \\
& = - \iint P_{ext} \partial_x(m\eta) dx dt - \int \partial_x(m\eta) \psi dx \Big|_0^T \\
& \quad - \iint (G(\eta)\psi)(m\psi_x) dx dt - \iint (N(\eta)\psi) \partial_x(m\eta) dx dt.
\end{aligned} \tag{3.10}$$

We then deduce the wanted result by splitting m_x as $1 + (m_x - 1)$ in the left-hand side. \square

The previous lemma gives an identity whose left hand-side is related to the integral in time of the potential energy. Since in the end we need to control the energy (and not only the potential energy), we need to compare this term to a similar one related to the kinetic energy. This is the purpose of the following lemma which, loosely speaking, is a version of the principle of equipartition of energy. Let us recall that the kinetic energy is given by

$$\frac{1}{2} \int_{-L}^L \int_{-h}^{\eta(t,x)} |\nabla_{x,y} \phi|^2 dy dx = \frac{1}{2} \int_{-L}^L \psi G(\eta) \psi dx,$$

where we used the divergence theorem and the fact that ϕ is harmonic.

Lemma 3.5. *For any smooth function $\theta = \theta(x)$, there holds*

$$\begin{aligned}
& \iint \theta \left(g\eta^2 + \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \right) dx dt = \iint \theta \psi G(\eta) \psi dx dt \\
& \quad - \iint \theta \eta P_{ext} dx dt - \int \theta \eta \psi dx \Big|_0^T \\
& \quad - \iint \theta \eta N(\eta) \psi dx dt \\
& \quad - \iint \kappa \theta_x \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} dx dt.
\end{aligned} \tag{3.11}$$

Proof. We have

$$\iint \theta \psi G(\eta) \psi dx dt = \iint \theta \psi \partial_t \eta dx dt = \int \theta \psi \eta dx \Big|_0^T - \iint \theta \eta \partial_t \psi dx dt.$$

So (3.11) easily follows from the equation for ψ (integrating by parts as above in the term involving the surface tension). \square

Recall that the energy is given at time t by

$$\mathcal{H}(t) = \frac{g}{2} \int \eta^2 dx + \kappa \int \left(\sqrt{1+\eta_x^2} - 1 \right) dx + \frac{1}{2} \int \psi G(\eta) \psi dx.$$

Instead of bounding $\int \mathcal{H} dt$, we will estimate the integral in time of the following quantity:

$$\tilde{H}(t) := \int \left(\frac{g}{2} \eta^2 + \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} + \frac{1}{2} \psi G(\eta) \psi \right) dx.$$

It will be sufficient to estimate this term since $\tilde{H}(t) \geq \mathcal{H}(t)$. Indeed,

$$\sqrt{1+\eta_x^2} - 1 = \frac{\eta_x^2}{1 + \sqrt{1+\eta_x^2}} \leq \frac{\eta_x^2}{\sqrt{1+\eta_x^2}}.$$

We now combine the previous results to obtain an identity for the integral in time of $\tilde{H}(t)$.

Lemma 3.6. *Set*

$$\zeta = \partial_x(m\eta) + \frac{3}{2}(1 - m_x)\eta - \frac{1}{4}\eta.$$

Then

$$\begin{aligned} \frac{1}{2} \int \tilde{H}(t) dt + \int R_2(t) dt &= \frac{3}{2} \iint (1 - m_x) \psi G(\eta) \psi dx dt \\ &\quad - \iint P_{ext} \zeta dx dt - \int \psi \zeta dx \Big|_0^T \\ &\quad - \iint m \psi_x G(\eta) \psi dx dt \\ &\quad - \iint \zeta N(\eta) \psi dx dt, \end{aligned} \quad (3.12)$$

where

$$\begin{aligned} R_2(t) &= -\frac{\kappa}{2} \int m_{xx} \frac{\eta \eta_x}{\sqrt{1 + \eta_x^2}} dx \\ &\quad + \kappa \int m_x \left(1 - \frac{\eta_x^2}{2\sqrt{1 + \eta_x^2}} - \frac{1}{\sqrt{1 + \eta_x^2}} \right) dx \\ &\quad + \frac{3\kappa}{4} \int \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} dx + g \int (1 - m_x) \eta^2 dx. \end{aligned}$$

Proof. We will deduce this identity from (3.7) and from Lemma 3.5. Denote by I_1 the first term of the left-hand side in (3.7), given by

$$I_1 = \iint \left(\frac{g}{2} \eta^2 + \frac{3}{2} \kappa \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} \right) dx dt.$$

We also introduce

$$I_m = \iint (1 - m_x) \left(\frac{g}{2} \eta^2 + \frac{3}{2} \kappa \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} \right) dx dt.$$

With these notations, the identity (3.7) reads

$$\begin{aligned} I_1 + \int R_1(t) dt &= I_m - \int_{-L}^L \partial_x(m\eta) \psi dx \Big|_0^T - \iint P_{ext} \partial_x(m\eta) dx dt \\ &\quad - \iint m \psi_x G(\eta) \psi dx dt - \iint \partial_x(m\eta) N(\eta) \psi dx dt. \end{aligned} \quad (3.13)$$

We split these terms as $I_1 = J_1 + P_1$ and $I_m = J_m + P_m$ with

$$\begin{aligned} J_1 &= \frac{1}{2} \iint \left(g \eta^2 + \kappa \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} \right) dx dt + \frac{\kappa}{4} \iint \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} dx dt, \\ P_1 &= \frac{3\kappa}{4} \iint \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} dx dt, \\ J_m &= \frac{3}{2} \iint (1 - m_x) \left(g \eta^2 + \kappa \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} \right) dx dt, \\ P_m &= - \iint (1 - m_x) g \eta^2 dx dt. \end{aligned}$$

We now use Lemma 3.5 to compute the first term contributing to J_1 . This gives

$$\begin{aligned} \iint \left(g\eta^2 + \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \right) dx dt &= \iint \psi G(\eta) \psi - \int \psi \eta dx \Big|_0^T \\ &\quad - \iint \eta N(\eta) \psi dx dt - \iint \eta P_{ext} dx dt. \end{aligned}$$

Consequently,

$$\begin{aligned} J_1 &= \frac{1}{2} \iint \left(\frac{g}{2} \eta^2 + \frac{1}{2} \kappa \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \right) dx dt + \frac{\kappa}{4} \iint \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} dx dt \\ &\quad + \frac{1}{4} \iint \psi G(\eta) \psi dx dt \\ &\quad - \frac{1}{4} \iint \eta P_{ext} dx dt - \frac{1}{4} \iint \eta N(\eta) \psi dx dt - \frac{1}{4} \int \psi \eta dx \Big|_0^T. \end{aligned}$$

Then, one checks that

$$J_1 = \frac{1}{2} \int \tilde{H} dt - \frac{1}{4} \iint \eta P_{ext} dx dt - \frac{1}{4} \iint \eta N(\eta) \psi dx dt - \frac{1}{4} \int \psi \eta dx \Big|_0^T.$$

We can also use Lemma 3.5 to obtain that

$$\begin{aligned} J_m &= \frac{3}{2} \iint (1 - m_x) \psi G(\eta) \psi dx dt \\ &\quad - \frac{3}{2} \iint (1 - m_x) \eta P_{ext} dx dt - \frac{3}{2} \iint (1 - m_x) \eta N(\eta) \psi dx dt \\ &\quad - \frac{3}{2} \int (1 - m_x) \psi \eta dx \Big|_0^T \\ &\quad + \frac{3}{2} \kappa \iint m_{xx} \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} dx dt. \end{aligned}$$

By combining the previous identities with (3.13) we obtain the wanted result (3.12). \square

Our next task is to study the last two terms in the right-hand side of (3.12),

$$\int m \psi_x G(\eta) \psi dx, \quad \int \zeta N(\eta) \psi dx.$$

The following lemma, which relies crucially on the results proved in [1], shows that these two terms can either be recast in a simpler way, or give rise to signed quantities (with the good signs).

Lemma 3.7. *Set*

$$\rho = \zeta + \eta - x \eta_x.$$

There holds

$$\begin{aligned} \frac{1}{2} \int \tilde{H}(t) dt + \int R_3(t) dt &= \frac{3}{2} \iint (1 - m_x) \psi G(\eta) \psi dx dt \\ &\quad - \iint P_{ext} \zeta dx dt - \int \psi \zeta dx \Big|_0^T \\ &\quad - \iint (m - x) \psi_x G(\eta) \psi dx dt \\ &\quad + \iiint \rho_x \phi_x \phi_y dy dx dt, \end{aligned} \tag{3.14}$$

where

$$R_3(t) = R_2(t) + \frac{1}{2} \int_{-L}^L (h + \rho) \phi_x^2|_{y=-h} dx + L \int_{-h}^{\eta(t,L)} \phi_y^2|_{x=L} dy.$$

Proof. To handle the term $\int m\psi_x G(\eta)\psi dx$ we split $m\psi_x$ as $m\psi_x = x\psi_x + (m-x)\psi_x$ to obtain an expression which makes appear a positive term through a Pohozaev identity. We will use the following identity (proved in [1])

$$\int_0^L (G(\eta)\psi)x\psi_x dx = \tilde{\Sigma} + \int_0^L (\eta - x\eta_x)N(\eta)\psi dx,$$

where $\tilde{\Sigma}(t)$ is a positive term given by

$$\tilde{\Sigma}(t) = \frac{h}{2} \int_0^L \phi_x^2(t, x, -h) dx + \frac{L}{2} \int_{-h}^{\eta(t,L)} \phi_y^2(t, L, y) dy.$$

Owing to the fact that η , ψ , ϕ_x^2 and ϕ_y^2 are even in x , we deduce that

$$\int_{-L}^L (G(\eta)\psi)x\psi_x dx = \Sigma + \int_{-L}^L (\eta - x\eta_x)(N(\eta)\psi) dx, \quad (3.15)$$

where

$$\Sigma(t) = \frac{h}{2} \int_{-L}^L \phi_x^2(t, x, -h) dx + L \int_{-h}^{\eta(t,L)} \phi_y^2(t, L, y) dy.$$

It follows from (3.15) that

$$\int_{-L}^L m\psi_x G(\eta)\psi dx = \Sigma(t) + \int_{-L}^L (\eta - x\eta_x)N(\eta)\psi dx + \int_{-L}^L (m-x)\psi_x G(\eta)\psi dx.$$

Recall that $\rho = \zeta + \eta - x\eta_x$. To complete the proof it remains only to show that one can write the integral $\int_{-L}^L \rho N(\eta)\psi dx$ as the sum of two terms which can be controlled by the sum of the energy and the positive term Σ given by the Pohozaev identity. To do so, we use the following identity (which is proved in the appendix, cf (A.3)),

$$\int_{-L}^L \rho N(\eta)\psi dx = - \int_{-L}^L \int_{-h}^{\eta(t,x)} \rho_x \phi_x \phi_y dy dx + \frac{1}{2} \int_{-L}^L \rho \phi_x^2|_{y=-h} dx. \quad (3.16)$$

By plugging this result into (3.12), we complete the proof of the lemma. \square

Recall that, by notation,

$$\tilde{\psi}(t, x) = \psi(t, x) - \langle \psi \rangle(t) \quad \text{with} \quad \langle \psi \rangle(t) = \frac{1}{2L} \int_{-L}^L \psi(t, x) dx.$$

We now have to prove that one can replace ψ by $\tilde{\psi}$ in the previous identity (3.14), to the price of an admissible error term.

Lemma 3.8. *There holds*

$$\begin{aligned} & \frac{1}{2} \int \tilde{H}(t) dt + \int R_4(t) dt \\ &= \iint \left(\frac{3}{2} (1 - m_x) \tilde{\psi} G(\eta) \psi + (x - m) \psi_x G(\eta) \psi \right) dx dt \\ & \quad - \iint P_{ext} \zeta dx dt - \int \tilde{\psi} \zeta dx \Big|_0^T \\ & \quad + \iiint \rho_x \phi_x \phi_y dy dx dt, \end{aligned} \quad (3.17)$$

where

$$R_4(t) = R_3(t) - \beta(t) \int \phi_x^2|_{y=-h} dx \quad \text{with} \quad \beta(t) = \frac{3}{8L} \int (1 - m_x) \eta dx.$$

Proof. Introduce

$$I = \iint \frac{3}{2} (1 - m_x) \langle \psi \rangle G(\eta) \psi dx dt - \int \langle \psi \rangle \zeta dx \Big|_0^T. \quad (3.18)$$

Then

$$\int R_3(t) dt - I = \int R_4(t) dt.$$

Since $\int \eta dx = 0$, by definition of ζ , we have

$$\int \zeta dx = \frac{3}{2} \int (1 - m_x) \eta dx.$$

Since $\partial_t \eta = G(\eta) \psi$, we deduce from (3.18) that

$$I = \iint \frac{3}{2} (1 - m_x) \langle \psi \rangle \partial_t \eta dx dt - \int \frac{3}{2} (1 - m_x) \eta \langle \psi \rangle dx \Big|_0^T.$$

Integrating by parts in time, this implies that

$$I = -\frac{3}{2} \iint (1 - m_x) \eta \partial_t \langle \psi \rangle dx dt,$$

and hence to compute I we need only to compute $\partial_t \langle \psi \rangle$. To do so, recall that

$$\partial_t \psi + g\eta + N(\eta)\psi - \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1 + \eta_x^2}} \right) + P_{ext} = 0.$$

Recall that $\int P_{ext} dx = 0$ and $\int \eta dx = 0$ by assumptions (see (2.4)). On the other hand, since η is C^1 and even in x , one has $\eta_x(t, -L) = \eta_x(t, L)$ and hence we conclude that $\langle \psi \rangle = (2L)^{-1} \int_{-L}^L \psi dx$ satisfies

$$\partial_t \langle \psi \rangle = -\frac{1}{2L} \int N(\eta) \psi dx.$$

Now we use that the identity (A.3) with $\mu = 1$, to obtain that

$$\partial_t \langle \psi \rangle = -\frac{1}{4L} \int \phi_x^2|_{y=-h} dx.$$

As a result,

$$I = \int \beta(t) \int \phi_x^2|_{y=-h} dx dt \quad \text{with} \quad \beta(t) = \frac{3}{8L} \int (1 - m_x) \eta dx.$$

This completes the proof of the lemma. \square

To complete the proof of the proposition, it remains to estimate the remainder terms. This is the purpose of the following lemma.

Lemma 3.9. *Assume that, for all time $t \in [0, T]$ and all $x \in [0, L]$, the following assumptions hold:*

- i) $\rho(t, x) \geq -\frac{h}{4}, \quad |\rho_x(t, x)| \leq \frac{1}{4},$
- ii) $\int_{-L}^L (1 - m_x(x)) \eta(t, x) dx \leq \frac{hL}{3}, \quad |m_x(x)| |\eta_x(t, x)|^2 \leq 2,$
- iii) $\kappa m_{xx}(x)^2 \leq g, \quad m_x(x) \leq 1.$

Then, for all $t \in [0, T]$,

$$R_4(t) - \iint_{\Omega(t)} \rho_x \phi_x \phi_y \, dy \, dx \geq \frac{h}{4} \int \phi_x^2|_{y=-h} \, dx - \frac{1}{4} \tilde{H}(t).$$

Proof. We have $R_4 = I_1 + \dots + I_6$ with

$$\begin{aligned} I_1 &= \int \left(\frac{h+\rho}{2} - \beta \right) \phi_x^2|_{y=-h} \, dx, & I_2 &= L \int \phi_y^2|_{x=L} \, dy, \\ I_3 &= \frac{3\kappa}{4} \int \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \, dx, & I_4 &= g \int (1-m_x) \eta^2 \, dx, \\ I_5 &= -\frac{\kappa}{2} \int m_{xx} \frac{\eta \eta_x}{\sqrt{1+\eta_x^2}} \, dx, \\ I_6 &= \kappa \int m_x \left(1 - \frac{\eta_x^2}{2\sqrt{1+\eta_x^2}} - \frac{1}{\sqrt{1+\eta_x^2}} \right) \, dx. \end{aligned}$$

By assumption *i*) and *ii*) we have $\rho \geq -h/4$ and $\beta \leq h/8$, so

$$I_1 \geq \frac{h}{4} \int \phi_x^2|_{y=-h} \, dx.$$

Obviously we have $I_2 \geq 0$ and $I_4 \geq 0$ since $m_x(x) \leq 1$ by assumption *iii*). So, to prove the lemma, we need only to show that

$$I_3 + I_5 + I_6 - \iint_{\Omega(t)} \rho_x \phi_x \phi_y \, dy \, dx \geq -\frac{1}{4} \tilde{H}(t). \quad (3.19)$$

We begin by estimating I_6 . Introduce $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ defined by

$$f(u) = 1 - \frac{u}{2\sqrt{1+u}} - \frac{1}{\sqrt{1+u}}.$$

Since $f(0) = 0$, we have the estimate

$$|f(u)| \leq \int_0^u |f'(s)| \, ds \leq \frac{1}{4} \int_0^u s \, ds = \frac{u^2}{8}.$$

So, since $|\eta_x| \leq 1$ by assumption, we may write

$$|I_6| \leq \kappa \int |m_x| \frac{\eta_x^4}{8} \, dx \leq \frac{\kappa}{4} \int |m_x| \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \eta_x^2 \, dx.$$

Now we use the assumption $|m_x(x)| |\eta_x(t, x)|^2 \leq 2$ to infer that

$$|I_6| \leq \frac{\kappa}{2} \int \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \, dx.$$

We now move to I_5 . Writing

$$\begin{aligned} |I_5| &\leq \frac{\kappa}{2} \int \frac{1}{4} m_{xx}^2 \frac{\eta^2}{\sqrt{1+\eta_x^2}} \, dx + \frac{\kappa}{2} \int \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \, dx \\ &\leq \frac{\kappa}{8} \sup_{[-L, L]} |m_{xx}|^2 \int \eta^2 \, dx + \frac{\kappa}{2} \int \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \, dx, \end{aligned}$$

and using assumption *iii*), one obtains that

$$|I_5| \leq \frac{g}{8} \int \eta^2 \, dx + \frac{\kappa}{2} \int \frac{\eta_x^2}{\sqrt{1+\eta_x^2}} \, dx.$$

On the other hand,

$$\begin{aligned} \iint_{\Omega(t)} \rho_x \phi_x \phi_y \, dy \, dx &\leq \frac{1}{2} \sup_{[-L, L]} |\rho_x| \iint_{\Omega(t)} (\phi_x^2 + \phi_y^2) \, dy \, dx \\ &\leq \frac{1}{8} \iint_{\Omega(t)} |\nabla_{x,y} \phi|^2 \, dy \, dx, \end{aligned}$$

since $|\rho_x| \leq 1/4$ by assumption. By combining the previous inequalities with the definition of I_3 , we conclude that the left-hand side of (3.19) is greater than

$$-\frac{g}{8} \int \eta^2 \, dx - \frac{\kappa}{4} \int \frac{\eta_x^2}{\sqrt{1 + \eta_x^2}} \, dx - \frac{1}{8} \iint_{\Omega(t)} |\nabla_{x,y} \phi|^2 \, dy \, dx = -\frac{1}{4} \tilde{H}(t).$$

This completes the proof of the lemma. \square

In view of Lemma 3.8 and Lemma 3.9, we obtain Proposition 3.1, recalling that $\tilde{H}(t) \geq \mathcal{H}(t)$ as explained above the statement of Lemma 3.6. \square

4. STABILIZATION

In this section we prove Theorem 2.6. In the previous section, we proved an inequality which is valid for any value of P_{ext} . Here, by contrast, we will prove an inequality which exploits in a crucial way some special choice for P_{ext} . We now assume that the external pressure is given by

$$P_{ext}(t, x) = P(t, x) + p(t),$$

where

$$P(t, x) = \lambda \chi(x) \partial_t \eta, \quad p(t) = -\frac{1}{2L} \int_{-L}^L P(t, x) \, dx,$$

for some positive constant λ and some cut-off function χ to be chosen. In the introduction, to state Theorem 2.6, we considered only the case $\lambda = 1$ (to fix ideas) and indeed we will not try to optimize λ . However, in this section we add this parameter for possible applications where it could be important to tune it.

Notice that, since $\partial_t \eta = G(\eta) \psi$, we have

$$P(t, x) = \lambda \chi(x) G(\eta) \psi.$$

Let us recall also how χ is defined.

Definition 4.1. Fix $\delta > 0$ and consider a $2L$ -periodic C^∞ function φ satisfying $0 \leq \varphi \leq 1$ and such that

$$\varphi(x) = \varphi(-x), \quad x\varphi'(x) \leq 0 \text{ for } x \in [-L, L], \quad \varphi(x) = \begin{cases} 1 & \text{if } x \in [0, L - \delta], \\ 0 & \text{if } x \in \left[L - \frac{\delta}{2}, L\right]. \end{cases}$$

We set

$$m(x) = x\varphi(x) \quad \text{and} \quad \chi(x) = 1 - m_x(x).$$

As already mentioned, it follows from the hamiltonian structure of the equation that

$$\frac{d}{dt} \mathcal{H}(t) = - \int_{-L}^L P_{ext} G(\eta) \psi \, dx.$$

In addition, since $\int_{-L}^L G(\eta) \psi \, dx = 0$, we get $\int_{-L}^L p(t) G(\eta) \psi \, dx = 0$ and hence

$$\frac{d}{dt} \mathcal{H}(t) = - \int_{-L}^L P G(\eta) \psi \, dx = -\lambda \int_{-L}^L \chi (G(\eta) \psi)^2 \, dx. \quad (4.1)$$

As a result, since $\lambda > 0$ and $\chi \geq 0$, we verify that the energy \mathcal{H} is a decreasing function of time. Our goal in this section is to obtain a quantitative result which asserts that, if the mild assumptions on η given by the statement of Theorem 2.6 are satisfied, then there exists a positive constant C , depending only on κ, g, h, L , such that

$$\mathcal{H}(T) \leq \frac{C}{T} \mathcal{H}(0). \quad (4.2)$$

To do so, as already explained in the introduction, we will prove that

$$\int_0^T \mathcal{H}(t) dt \leq C \mathcal{H}(0). \quad (4.3)$$

This will imply the desired result (4.2), by writing

$$\mathcal{H}(T) \leq \frac{1}{T} \int_0^T \mathcal{H}(t) dt \leq \frac{C}{T} \mathcal{H}(0).$$

We begin by establishing two properties of the pressure which explain the choices of the expressions for P , P_{ext} and χ .

Lemma 4.2. *There holds*

$$\int_0^T \int_{-L}^L P^2 dx dt \leq \lambda \left(\sup_{[-L,L]} \chi \right) \mathcal{H}(0), \quad (4.4)$$

and

$$- \int_0^T p(t) \int_{-L}^L \zeta(t, x) dx dt \leq \frac{3\lambda}{2g} \sup_{[-L,L]} (1 - m_x)^2 \mathcal{H}(0), \quad (4.5)$$

where recall that ζ is given by

$$\zeta = \partial_x(m\eta) + \frac{3}{2}(1 - m_x)\eta - \frac{1}{4}\eta.$$

Proof. Firstly, integrating in time the identity (4.1), we obtain

$$\lambda \int_0^T \int_{-L}^L \chi (G(\eta)\psi)^2 dx dt = \mathcal{H}(0) - \mathcal{H}(T) \leq \mathcal{H}(0). \quad (4.6)$$

This immediately implies that

$$\int_0^T \int_{-L}^L P^2 dx dt = \lambda^2 \int_0^T \int_{-L}^L \chi^2 (G(\eta)\psi)^2 dx dt \leq \lambda (\sup \chi) \mathcal{H}(0),$$

which is (4.4).

To estimate $\int \int p \zeta dx dt$ we use in a crucial way the fact that, by definition, $\chi(x) = 1 - m_x(x)$. Since $\int \eta dx = 0$, directly from the definition of ζ , we have

$$\int_{-L}^L \zeta(t, x) dx = \frac{3}{2} \nu(t) \quad \text{where} \quad \nu(t) := \int_{-L}^L (1 - m_x) \eta dx.$$

So

$$\int_{-L}^L p(t) \zeta(t, x) dx = \frac{3}{2} p(t) \nu(t).$$

On the other hand, by definition,

$$p = -\frac{\lambda}{2L} \int_{-L}^L \chi G(\eta) \psi dx = -\frac{\lambda}{2L} \int_{-L}^L \chi \partial_t \eta dx = -\frac{\lambda}{2L} \partial_t \int_{-L}^L \chi \eta dx.$$

Since $\chi = 1 - m_x$, we deduce that

$$p = -\frac{\lambda}{2L} \partial_t \nu,$$

and hence

$$\int_0^T p(t) \int_{-L}^L \zeta(t, x) dx dt = -\frac{3\lambda}{4L} \int_0^T \nu \partial_t \nu dt.$$

Consequently,

$$-\int_0^T \int_{-L}^L p(t) \zeta(t, x) dx dt = \frac{3\lambda}{8L} \int_0^T \frac{d}{dt} \nu^2 dt \leq \frac{3\lambda}{8L} \nu(T)^2.$$

Now, to estimate $\nu(T)^2$ we use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \nu(T)^2 &= \left(\int_{-L}^L (1 - m_x) \eta dx \right)^2 \leq 2L \int_{-L}^L (1 - m_x)^2 \eta^2 dx \\ &\leq 2L \sup_{[-L, L]} (1 - m_x)^2 \int_{-L}^L \eta^2 dx \\ &\leq \frac{4L}{g} \sup_{[-L, L]} (1 - m_x)^2 \mathcal{H}(T) \leq \frac{4L}{g} \sup_{[-L, L]} (1 - m_x)^2 \mathcal{H}(0), \end{aligned}$$

since \mathcal{H} is decreasing. This yields

$$-\int_0^T \int_{-L}^L p(t) \zeta(t, x) dx dt \leq \frac{3\lambda}{2g} \sup_{[-L, L]} (1 - m_x)^2 \mathcal{H}(0),$$

which completes the proof of the lemma. \square

We now observe that the assumptions of Theorem 2.6 are stronger than the ones of Proposition 3.1 so we may apply this proposition to write that

$$\frac{1}{4} \int_0^T \mathcal{H}(t) dt \leq O + W + B - I, \quad (4.7)$$

where

$$\begin{aligned} I &:= \frac{h}{4} \int_0^T \int_{-L}^L \phi_x(t, x, -h)^2 dx dt, \\ O &:= \int_0^T \int_{-L}^L \left(\frac{3}{2} (1 - m_x) \tilde{\psi} + (x - m) \psi_x \right) G(\eta) \psi dx dt, \\ W &:= - \int_0^T \int_{-L}^L P_{ext} \zeta dx dt, \\ B &:= - \int_{-L}^L \zeta \tilde{\psi} dx \Big|_0^T. \end{aligned}$$

To deduce (4.3) from (4.7), as already explained, it is sufficient to prove that there exists a constant K depending only on g, κ, h, L such that

$$O + W + B - I \leq K \mathcal{H}(0) + a \int_0^T \mathcal{H}(t) dt \quad \text{for some } a < \frac{1}{4}. \quad (4.8)$$

Indeed, (4.8) implies

$$\int_0^T \mathcal{H}(t) dt \leq \frac{K}{1/4 - a} \mathcal{H}(0),$$

which is the wanted result (4.3).

To prove (4.8), we begin by estimating the term W .

Lemma 4.3. *Let $\varepsilon_1 > 0$ and set*

$$C_1 = \frac{\lambda}{2\varepsilon_1} \sup_{[-L,L]} \chi(x) + \frac{3\lambda}{2g} \sup_{[-L,L]} (1 - m_x(x))^2, \quad (4.9)$$

$$K_1 = \frac{6}{\kappa} \sup_{[-L,L]} m(x)^2 + \frac{4}{g} \sup_{[-L,L]} \left(\frac{5}{4} - \frac{m_x(x)}{2} \right)^2. \quad (4.10)$$

Then there holds

$$- \iint P_{ext} \zeta \, dx \, dt \leq C_1 \mathcal{H}(0) + \frac{\varepsilon_1 K_1}{2} \int \mathcal{H}(t) \, dt. \quad (4.11)$$

Proof. Recall that $P = P_{ext} + p$. We have already estimated $-\iint p\zeta \, dx \, dt$ so to prove (4.11) it remains only to estimate $-\iint P\zeta \, dx \, dt$. Firstly, using (4.4) and the Young's inequality $-ab \leq (1/2\varepsilon_1)a^2 + (\varepsilon_1/2)b^2$, we have

$$\begin{aligned} - \iint P\zeta \, dx \, dt &\leq \frac{1}{2\varepsilon_1} \iint P^2 \, dx \, dt + \frac{\varepsilon_1}{2} \iint \zeta^2 \, dx \, dt \\ &\leq \frac{\lambda}{2\varepsilon_1} (\sup \chi) \mathcal{H}(0) + \frac{\varepsilon_1}{2} \iint \zeta^2 \, dx \, dt. \end{aligned}$$

This implies that $-\iint P\zeta \, dx \, dt$ is bounded by the right-hand side of (4.11) since

$$\int \zeta(t, x)^2 \, dx \leq K_1 \mathcal{H}(t), \quad (4.12)$$

by definition of ζ , K_1 and \mathcal{H} . Indeed,

$$\begin{aligned} \int \zeta^2 \, dx &\leq \int 2m^2 \eta_x^2 \, dx + \int 2 \left(\frac{5}{4} - \frac{m_x}{2} \right)^2 \eta^2 \, dx \\ &\leq \frac{6m^2}{\kappa} \int \kappa \left(\sqrt{1 + \eta_x^2} - 1 \right) \, dx + \frac{4}{g} \int \left(\frac{5}{4} - \frac{m_x}{2} \right)^2 \frac{g}{2} \eta^2 \, dx, \end{aligned}$$

where we used that $1 + \sqrt{1 + \eta_x^2} \leq 3$ since $|\eta_x| \leq 1$ by assumption. \square

We now estimate the term B in (4.7).

Lemma 4.4. *There exists a constant K_2 depending only on h, L such that,*

$$\int_{-L}^L \tilde{\psi}(t, x)^2 \, dx \leq K_2 \mathcal{H}(t). \quad (4.13)$$

Moreover,

$$\left| \int_{-L}^L \zeta \tilde{\psi} \, dx \right|_0^T \leq (K_1 + K_2) \mathcal{H}(0), \quad (4.14)$$

where K_1 is given by (4.10).

Proof. The first estimate follows from usual estimates (Poincaré's inequality) and we postpone its proof to Appendix B. To prove (4.14), consider $t_0 = 0$ or $t_0 = T$ and write

$$\begin{aligned} \left| \int \zeta(t_0, x) \tilde{\psi}(t_0, x) \, dx \right| &\leq \frac{1}{2} \int \zeta^2(t_0, x) \, dx + \frac{1}{2} \int \tilde{\psi}^2(t_0, x) \, dx \\ &\leq \frac{1}{2} (K_1 + K_2) \mathcal{H}(t_0) \leq \frac{1}{2} (K_1 + K_2) \mathcal{H}(0), \end{aligned}$$

where we used (4.12) and the fact that \mathcal{H} is decreasing. This implies (4.14). \square

We are now in position to estimate the last integral, which is

$$O = \int_0^T \int_{-L}^L \left(\frac{3}{2}(1 - m_x)\tilde{\psi} + (x - m)\psi_x \right) G(\eta)\psi \, dx \, dt.$$

Lemma 4.5. *For any positive real numbers $\varepsilon_2, \varepsilon_3$, there holds*

$$\begin{aligned} |O| &\leq \left(\frac{1}{\varepsilon_2 \lambda} \sup_{[-L, L]} \chi + \frac{4\varepsilon_3 L}{\lambda} + \frac{L}{2\varepsilon_3 \lambda} \right) \mathcal{H}(0) + 2\varepsilon_3 L \int_{-L}^L \phi_x^2|_{y=-h} \, dx \, dt \\ &\quad + \left(\frac{9\varepsilon_2}{16} K_2 + 4\varepsilon_3 L \sup |\chi_x| \right) \int \mathcal{H}(t) \, dt. \end{aligned}$$

Proof. We split O as $O_1 + O_2$ with

$$O_1 = \iint \frac{3}{2}(1 - m_x)\tilde{\psi} G(\eta)\psi \, dx \, dt, \quad O_2 = \iint (x - m)\psi_x G(\eta)\psi \, dx \, dt.$$

Since $\chi = 1 - m_x$, for any $\varepsilon_2 > 0$, one has the estimate

$$\begin{aligned} |O_1| &\leq \frac{9\varepsilon_2}{16} \iint \tilde{\psi}^2 \, dx \, dt + \frac{1}{\varepsilon_2} \iint (1 - m_x)^2 (G(\eta)\psi)^2 \, dx \, dt \\ &\leq \frac{9\varepsilon_2}{16} \iint \tilde{\psi}^2 \, dx \, dt + \frac{1}{\varepsilon_2} \sup_{[-L, L]} \chi \iint \chi (G(\eta)\psi)^2 \, dx \, dt. \end{aligned}$$

An application of (4.6) gives the estimate

$$|O_1| \leq \frac{9\varepsilon_2}{16} \iint \tilde{\psi}^2 \, dx \, dt + \frac{1}{\varepsilon_2 \lambda} \left(\sup_{[-L, L]} \chi \right) \mathcal{H}(0).$$

It follows from (4.13) that

$$|O_1| \leq \frac{9\varepsilon_2}{16} K_2 \int \mathcal{H}(t) \, dt + \frac{1}{\varepsilon_2 \lambda} \left(\sup_{[-L, L]} \chi \right) \mathcal{H}(0).$$

The estimate of O_2 is more delicate. We begin by proving that, for any $x \in [-L, L]$, one has

$$|x - m(x)| \leq L\chi(x). \quad (4.15)$$

To see this, recall that m is as defined in Definition 2.4. Since m is odd in x , we can assume without loss of generality that $x \in [0, L]$, so that

$$x - m(x) = x(1 - \varphi(x)) \leq L(1 - \varphi(x)) \leq L(1 - \varphi(x) - x\varphi'(x)) = L(1 - m_x(x)),$$

where we used again the definition $\chi(x) = 1 - m_x(x)$ together with the assumptions $\varphi(x) \leq 1$ and $x\varphi'(x) \leq 0$.

An application of (4.15) and the Young's inequality implies that, for any $\varepsilon_3 > 0$,

$$|(x - m)\psi_x G(\eta)\psi| \leq \frac{\varepsilon_3 L}{2} \chi \psi_x^2 + \frac{L}{2\varepsilon_3} \chi (G(\eta)\psi)^2.$$

Now we claim that if $|\eta_x| \leq 1$ then

$$\begin{aligned} \int_{-L}^L \chi \psi_x^2 \, dx &\leq 8 \int_{-L}^L \chi (G(\eta)\psi)^2 \, dx + 4 \int_{-L}^L \phi_x^2|_{y=-h} \, dx \\ &\quad - 8 \int_{-L}^L \int_{-h}^{\eta(t, x)} \chi_x \phi_x \phi_y \, dy \, dx. \end{aligned} \quad (4.16)$$

Assume that this is proved. Then an application of the estimate

$$\iint \chi (G(\eta)\psi)^2 \, dx \, dt \leq \frac{1}{\lambda} \mathcal{H}(0),$$

will imply that

$$|O_2| \leq \frac{1}{\lambda} \left(\frac{\varepsilon_3 L}{2} 8 + \frac{L}{2\varepsilon_3} \right) \mathcal{H}(0) + \frac{\varepsilon_3 L}{2} 4 \int_{-L}^L \phi_x^2|_{y=-h} dx dt \\ - \frac{\varepsilon_3 L}{2} 8 \iiint \chi_x \phi_x \phi_y dy dx dt,$$

so

$$|O_2| \leq \left(\frac{4\varepsilon_3 L}{\lambda} + \frac{L}{2\varepsilon_3 \lambda} \right) \mathcal{H}(0) + 2\varepsilon_3 L \int_{-L}^L \phi_x^2|_{y=-h} dx dt \\ + 4\varepsilon_3 L \sup |\chi_x| \iiint \frac{1}{2} |\nabla_{x,y} \phi|^2 dy dx dt,$$

and hence

$$|O_2| \leq \left(\frac{4\varepsilon_3 L}{\lambda} + \frac{L}{2\varepsilon_3 \lambda} \right) \mathcal{H}(0) + 2\varepsilon_3 L \int_{-L}^L \phi_x^2|_{y=-h} dx dt \\ + 4\varepsilon_3 L \sup |\chi_x| \int \mathcal{H}(t) dt.$$

It remains to prove (4.16). Introduce the notations $V = \phi_x|_{y=\eta}$, $B = \phi_y|_{y=\eta}$ and recall that, by definition, $G(\eta)\psi = \phi_y|_{y=\eta} - \eta_x \phi_x|_{y=\eta} = B - \eta_x V$. Consequently, one has the identity

$$(G(\eta)\psi)^2 = B^2 - 2\eta_x BV + \eta_x^2 V^2 \\ = B^2 - V^2 - 2\eta_x BV + (1 + \eta_x^2)V^2.$$

Now, in terms of $N(\eta)\psi$ (see (2.1)), this gives

$$(1 + \eta_x^2)V^2 = (G(\eta)\psi)^2 + 2N(\eta)\psi.$$

We next use that (cf (A.3))

$$\int_{-L}^L \chi N(\eta)\psi dx = - \iint_{\Omega} \chi_x \phi_x \phi_y dy dx + \frac{1}{2} \int_{-L}^L \chi \phi_x^2|_{y=-h} dx,$$

to infer that

$$\int_{-L}^L \chi (1 + \eta_x^2)V^2 dx = \int_{-L}^L \chi (G(\eta)\psi)^2 dx + \int_{-L}^L \chi \phi_x^2|_{y=-h} dx \\ - 2 \int_{-L}^L \int_{-h}^{\eta(t,x)} \chi_x \phi_x \phi_y dy dx. \quad (4.17)$$

To conclude the proof of (4.16), it remains to estimate $\int_{-L}^L \chi (1 + \eta_x^2)\psi_x^2 dx$ by means of $\int_{-L}^L \chi (1 + \eta_x^2)V^2 dx$. To do so, we start from the following identities (which easily follow from the definitions of V, B)

$$V = \psi_x - \eta_x B, \quad B = G(\eta)\psi + \eta_x V,$$

to obtain, using the assumption $|\eta_x| \leq 1$,

$$\psi_x^2 = (V + \eta_x B)^2 \leq 2V^2 + 2\eta_x^2 B^2 \\ \leq 2V^2 + 4\eta_x^2 (G(\eta)\psi)^2 + 4\eta_x^4 V^2 \\ \leq 4(1 + \eta_x^2)V^2 + 4(G(\eta)\psi)^2.$$

Then the wanted inequality (4.16) follows from (4.17). \square

Now, by combining the previous estimates, we conclude that

$$\begin{aligned} O + W + B - I &\leq \left(C_1 + K_1 + K_2 + \frac{1}{\varepsilon_2 \lambda} \sup_{[-L, L]} \chi + \frac{4\varepsilon_3 L}{\lambda} + \frac{L}{2\varepsilon_3 \lambda} \right) \mathcal{H}(0) \\ &\quad + \left(2\varepsilon_3 L - \frac{h}{4} \right) \int_{-L}^L \phi_x^2|_{y=-h} dx dt \\ &\quad + \left(\frac{\varepsilon_1 K_1}{2} + \frac{9\varepsilon_2}{16} K_2 + 4\varepsilon_3 L \sup |\chi_x| \right) \int \mathcal{H}(t) dt, \end{aligned}$$

where

$$\begin{aligned} C_1 &= \frac{\lambda}{2\varepsilon_1} \sup_{[-L, L]} \chi(x) + \frac{3\lambda}{2g} \sup_{[-L, L]} (1 - m_x(x))^2, \\ K_1 &= \frac{6}{\kappa} \sup_{[-L, L]} m(x)^2 + \frac{4}{g} \sup_{[-L, L]} \left(\frac{5}{4} - \frac{m_x(x)}{2} \right)^2, \end{aligned}$$

and where K_2 is given by (4.13).

Now, we fix $\lambda = 1$ (we do not consider the problem of optimizing λ) and we fix $\varepsilon_1, \varepsilon_2$ and ε_3 small enough, so that

$$2\varepsilon_3 L - \frac{h}{4} \leq 0, \quad \frac{\varepsilon_1 K_1}{2} + \frac{9\varepsilon_2}{16} K_2 + 4\varepsilon_3 L \sup |\chi_x| \leq \frac{1}{8}.$$

Then

$$O + W + B - I \leq K \mathcal{H}(0) + \frac{1}{8} \int_0^T \mathcal{H}(t) dt, \quad (4.18)$$

where K is defined by

$$K := C_1 + K_1 + K_2 + \frac{1}{\varepsilon_2 \lambda} \sup_{[-L, L]} \chi + \frac{4\varepsilon_3 L}{\lambda} + \frac{L}{2\varepsilon_3 \lambda}. \quad (4.19)$$

Notice that K depends only on κ, g, L, h . Then we obtain (4.3) with $C = 8K$. This concludes the proof of Theorem 2.6.

APPENDIX A.

For the sake of completeness, we recall from [2] the proof of the identity (3.16).

Here the time variable is seen as a parameter and we drop it. We consider a real number $s > 5/2$ and two $2L$ -periodic functions $\eta, \psi \in H^s(\mathbb{T})$ which are even in x . We denote by ϕ the harmonic function defined by

$$\begin{cases} \Delta_{x,y} \phi = 0 & \text{in } \Omega = \{(x, y) \in \mathbb{T} \times \mathbb{R}; -h < y < \eta(x)\}, \\ \phi(x, \eta(x)) = \psi(x), \\ \phi_y(x, -h) = 0. \end{cases} \quad (A.1)$$

It follows that $\nabla_{x,y} \phi$ belongs to $C^1(\overline{\Omega})$ and that $\phi_x = 0$ when $x = -L$ or $x = L$ (see Proposition 2.2 in [1]). As a result one can take the trace of $\nabla_{x,y} \phi$ and define

$$N(\eta)\psi = \mathcal{N}|_{y=\eta} \quad \text{with} \quad \mathcal{N} = \frac{1}{2}\phi_x^2 - \frac{1}{2}\phi_y^2 + \eta_x \phi_x \phi_y. \quad (A.2)$$

Lemma A.1. *For any C^1 function $\mu = \mu(x)$ which is $2L$ -periodic, there holds*

$$\int_{-L}^L \mu N(\eta)\psi dx = - \iint_{\Omega} \mu_x \phi_x \phi_y dy dx + \frac{1}{2} \int_{-L}^L \mu \phi_x^2|_{y=-h} dx. \quad (A.3)$$

Proof. The proof relies on the following identity

$$\partial_y(\phi_y^2 - \phi_x^2) + 2\partial_x(\phi_x\phi_y) = 2\phi_y\Delta_{x,y}\phi,$$

which implies that, since ϕ is harmonic and $\partial_y\mu = 0$,

$$\partial_y(\mu\phi_y^2 - \mu\phi_x^2) + 2\partial_x(\mu\phi_x\phi_y) = 2\mu_x\phi_x\phi_y.$$

We deduce that the vector field $X: \Omega \rightarrow \mathbb{R}^2$ defined by $X = (-\mu\phi_x\phi_y; \frac{\mu}{2}\phi_x^2 - \frac{\mu}{2}\phi_y^2)$ satisfies $\operatorname{div}_{x,y}(X) = -\mu_x\phi_x\phi_y$. Since $\nabla_{x,y}\phi$ belongs to $C^1(\overline{\Omega})$ and since one has the boundary conditions

$$\phi_y|_{y=-h} = \phi_x|_{x=-L} = \phi_x|_{x=L} = 0,$$

an application of the divergence theorem gives that

$$\begin{aligned} -\iint_{\Omega} \mu_x\phi_x\phi_y \, dy \, dx &= \iint_{\Omega} \operatorname{div}_{x,y} X \, dy \, dx \\ &= \int_{\partial\Omega} X \cdot n \, d\sigma = \int_{-L}^L \mu \mathcal{N}|_{y=\eta} \, dx - \frac{1}{2} \int_{-L}^L \mu\phi_x^2|_{y=-h} \, dx. \end{aligned}$$

This completes the proof. \square

APPENDIX B.

Let us prove (4.13). The time variable is seen as a parameter and we drop it. As above, we introduce the harmonic extension of $\tilde{\psi}$ defined by

$$\begin{cases} \Delta_{x,y}\tilde{\phi} = 0 & \text{in } \Omega = \{(x,y) \in \mathbb{T} \times \mathbb{R}; -h < y < \eta(x)\}, \\ \tilde{\phi}(x, \eta(x)) = \tilde{\psi}(x), \\ \tilde{\phi}_y(x, -h) = 0. \end{cases}$$

Since $\tilde{\psi} = \psi - \langle \psi \rangle$ where $\langle \psi \rangle = \frac{1}{2L} \int_{-L}^L \psi(x) \, dx$ is a constant, we have $\tilde{\phi} = \phi - \langle \psi \rangle$ where ϕ is the harmonic extension of ψ .

It will be useful to consider a diffeomorphism Σ from the flat strip $\mathbb{T} \times [-1, 0]$ to the domain Ω , of the form

$$\Sigma: (x, z) \mapsto (x, \sigma(x, z)) \quad \text{with } \sigma(x, z) = (1+z)\eta(x) + hz.$$

Since $\eta \geq -h/2$ by assumption, we easily verify that Σ is a diffeomorphism from $\mathbb{T} \times [-1, 0]$ to the domain Ω . Then we set

$$\tilde{\varphi}(x, z) = \tilde{\phi}(x, \sigma(x, z)).$$

Recall that $|\eta_x(t, x)| \leq 1$ by assumption (cf Theorem 2.6). Then, directly from the change of variables formula for integrals, we obtain that there exists a constant C depending only on h , such that

$$\|\nabla_{x,z}\tilde{\varphi}\|_{L^2(\mathbb{T} \times [-1, 0])}^2 \leq C \|\nabla_{x,y}\tilde{\phi}\|_{L^2(\Omega)}^2 = C \|\nabla_{x,y}\phi\|_{L^2(\Omega)}^2 \leq C\mathcal{H}.$$

Consequently, to prove (4.13) it is sufficient to prove that

$$\|\tilde{\psi}\|_{L^2}^2 \leq C' \|\nabla_{x,z}\tilde{\varphi}\|_{L^2(\mathbb{T} \times [-1, 0])}^2, \tag{B.1}$$

for some constant C' depending only on h, L . To do so, we proceed in two steps. Firstly, we set

$$c_n := \frac{1}{2L} \int_{-L}^L \tilde{\psi}(x) e^{-\frac{\pi}{L}inx} \, dx$$

and use the fact that $\tilde{\psi}$ is $2L$ -periodic to obtain

$$\|\tilde{\psi}\|_{L^2(\mathbb{T})}^2 = 2L \sum_{n \in \mathbb{Z}} |c_n|^2.$$

By assumption, the mean of $\tilde{\psi}$ vanishes and hence $c_0 = 0$. This immediately implies that

$$\|\tilde{\psi}\|_{L^2(\mathbb{T})}^2 \leq 2L \sum_{n \in \mathbb{Z}} |n| |c_n|^2 \leq A \| |D_x|^{1/2} \tilde{\psi} \|_{L^2}^2,$$

where the constant A depends only on L and $|D_x|^{1/2}$ is the Fourier multiplier with symbol $|\xi|^{1/2}$. Now, to bound $\| |D_x|^{1/2} \tilde{\psi} \|_{L^2}^2$ by $\| \nabla_{x,z} \tilde{\varphi} \|_{L^2(\mathbb{T} \times [-1,0])}^2$, we may proceed as in the proof of Proposition 3.12 in [23].

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